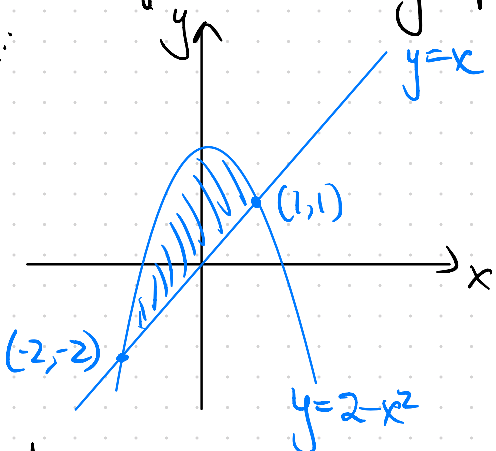
14.2 66) Find the volume of the solid that is bounded above by the cylinder $z=x^2$ and below by the region enclosed by the parabola $y=2-x^2$ and the line $y=x$ in the xy-plane.

Sol'n:



$$V=\int_{-2}^{1}\int_{x}^{2-x^2} x^2\,dy\,dx = \int_{-2}^{1} x^2 y\Big|_{y=x}^{y=2-x^2} dx$$

$$=\int_{-2}^{1}\left(x^2(2-x^2)-x^2(x)\right)dx$$

$$=\int_{-2}^{1}(2x^2-x^4-x^3)\,dx = \frac{2}{3}x^3\Big|_{-2}^{1}-\frac{1}{5}x^5\Big|_{-2}^{1}-\frac{1}{4}x^4\Big|_{-2}^{1}$$

$$=\frac{2}{3}(1-(-8))-\frac{1}{5}(1-(-32))-\frac{1}{4}(1-(16))$$

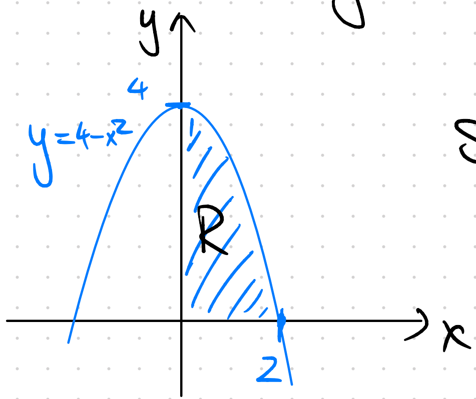
$$=\boxed{\frac{63}{20}}$$

14.2 70) Find the volume of the solid cut ~~by~~ from the first octant by the surface

$$z = 4 - x^2 - y$$

<u>Sol'n</u>: 1^{st} octant, $z \geq 0$, $y \geq 0$, $x \geq 0$. Project onto xy-plane: $z = 0 \Rightarrow 0 = 4 - x^2 - y$

$\Rightarrow y = 4 - x^2$.



So $V = \iint_R z\,dA = \int_0^2 \int_0^{4-x^2} (4 - x^2 - y)\,dy\,dx$

$$= \int_0^2 \left((4-x^2)y - \tfrac{1}{2}y^2\right)\Big|_{y=0}^{y=4-x^2} dx$$

$$= \int_0^2 (4-x^2)^2 - \tfrac{1}{2}(4-x^2)^2\,dx = \tfrac{1}{2}\int_0^2 (4-x^2)^2\,dx = \tfrac{1}{2}\int_0^2 (16 - 8x^2 + x^4)\,dx$$

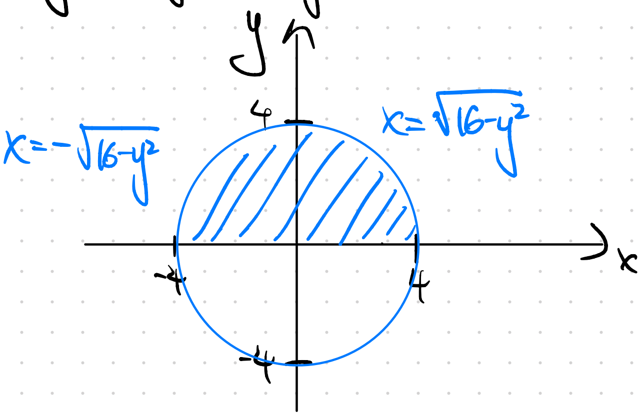
$$= \tfrac{1}{2}\left(16x\Big|_0^2 - \tfrac{8}{3}x^3\Big|_0^2 + \tfrac{1}{5}x^5\Big|_0^2\right) = \boxed{\frac{128}{5}}$$

14.2 76) Sketch the region of integration and the solid whose volume is given by the double integral:

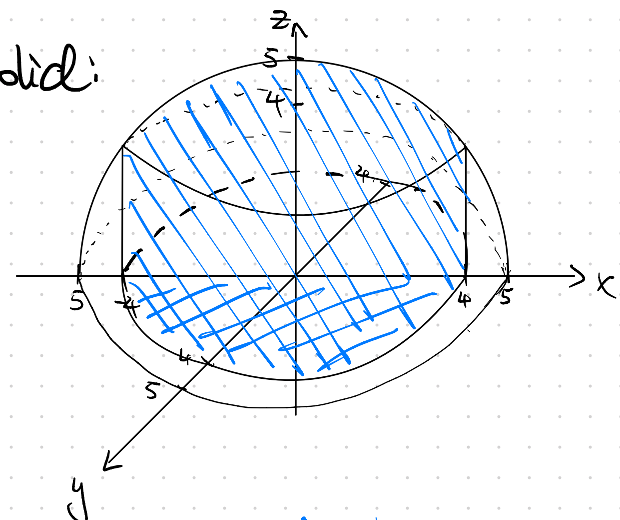
$$\int_0^4 \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \sqrt{25-x^2-y^2}\,dx\,dy$$

Sol'n:

Region of Integration:



Solid:

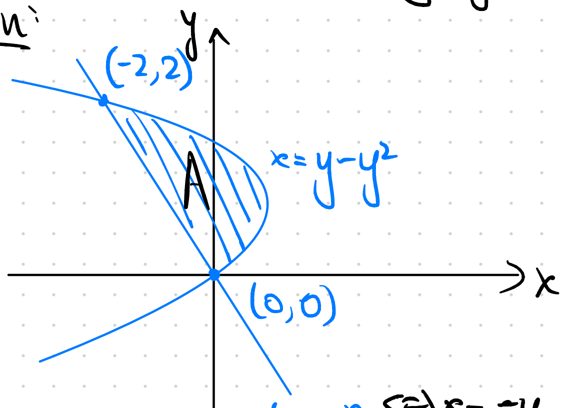


Solid is shaded in blue

14.3 4) Sketch the region bounded by the given lines and curves. Then express the region's area as an iterated double integral and evaluate the integral:

- The parabola $x=y-y^2$ and the line $y=-x$

Sol'n:



$y=-x \Leftrightarrow x=-y$

Intersection pts: $x=-x-(-x)^2$

$=-x-x^2$

$x^2+2x=0$

$x(x-2)=0$

$\Rightarrow x=0,\ -2.$

$$A=\int_0^2\int_{-y}^{y-y^2} dx\,dy=\int_0^2 x\Big|_{x=-y}^{x=y-y^2} dy$$

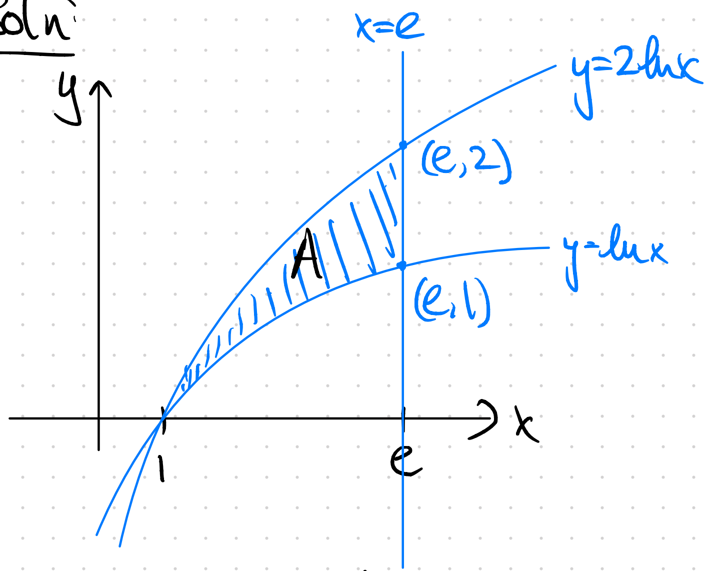
$$=\int_0^2\left(y-y^2-(-y)\right)dy=\int_0^2(2y-y^2)\,dy$$

$$=y^2\Big|_0^2-\frac{1}{3}y^3\Big|_0^2=4-\frac{8}{3}=\boxed{\frac{4}{3}}$$

14.3 6) Sketch the region bounded by the given lines and curves. Then express the region's area as an iterated double integral and evaluate the integral:

- The curves $y = \ln x$ and $y = 2\ln x$ and the line $x = e$, in the first quadrant.

Sol'n:



Intersection pts: $y_1 = \ln(e) = 1$

$y_2 = 2\ln(e) = 2.$

$$A = \int_1^e \int_{\ln x}^{2\ln x} dy\,dx = \int_1^e y\Big|_{y=\ln x}^{y=2\ln x} dx$$

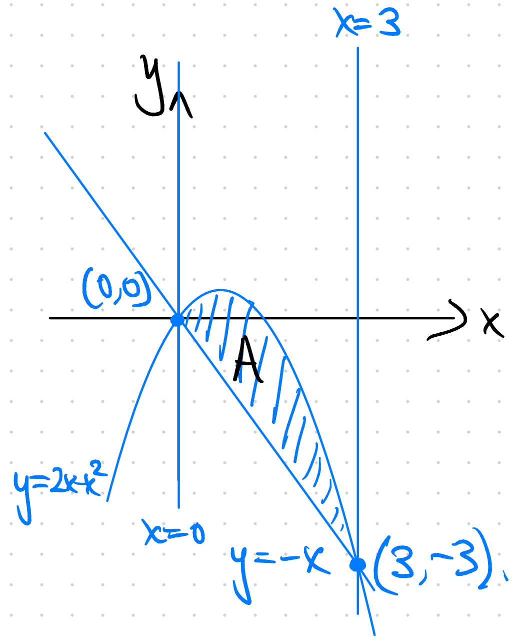
$$= \int_1^e (2\ln x - \ln x)\,dx = \int_1^e \ln x\,dx$$

$$= (x\ln x - x)\Big|_1^e = e\ln(e) - e - (\overset{0}{\cancel{1\ln(1)}} - 1)$$

$$= \cancel{e} - \cancel{e} - (-1) = \boxed{1}$$

14.3 14) Sketch the region, label each bounding curve with its equation, and give the coordinates of the points where the curves intersect. Then find the area of the region:

$$\int_0^3 \int_{-x}^{x(2-x)} dy\,dx.$$



Sol'n: Line 1 : $y = -x$

Line 2 : $y = x(2-x) = 2x - x^2$.

Line 3: $x = 0$

Line 4: $x = 3$.

Intersection points:

Line 3 w/ Line 1 and Line 2: $(x=0, y=0)$

Line 4 w/ Line 1 : $(x=3, y=-3)$

Line 4 w/ Line 2: $y = 2(3) - (3)^2 = 6 - 9 = -3$. : $(3,-3)$

Line 1 w/ Line 2: $-x = 2x - x^2 \Leftrightarrow 0 = 3x - x^2 = x(3-x) \Rightarrow$ $(x=0, y=0)$ $(x=3, y=-3)$.

$$A = \int_0^3 \int_{-x}^{x(2-x)} dy\,dx = \int_0^3 y\Big|_{y=-x}^{y=x(2-x)} dx = \int_0^3 (x(2-x)-(-x))\,dx$$

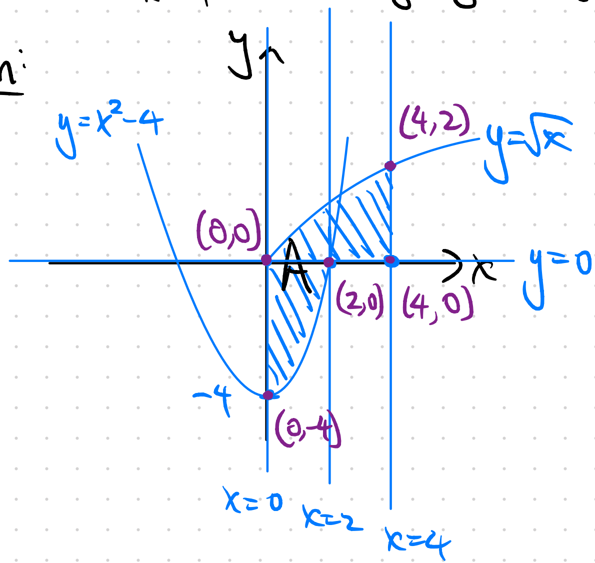
$$= \int_0^3 (2x - x^2 + x)\,dx = \int_0^3 (3x - x^2)\,dx = \frac{3}{2}x^2\Big|_0^3 - \frac{1}{3}x^3\Big|_0^3$$

$$= \frac{27}{2} - \frac{27}{3} = \boxed{\frac{9}{2}}$$

14.3 18) Sketch the region, label each bounding curve with its equation, and give the coordinates of the points where the curves intersect. Then find the area of the region:

$$\int_0^2\int_{x^2-4}^0 dy\,dx + \int_0^4\int_0^{\sqrt{x}} dy\,dx$$

Sol'n:



$$A=\int_0^2\int_{x^2-4}^0 dy\,dx+\int_0^4\int_0^{\sqrt{x}} dy\,dx$$

$$=\int_0^2 y\Big|_{y=x^2-4}^{y=0} dx + \int_0^4 y\Big|_{y=0}^{y=\sqrt{x}} dx$$

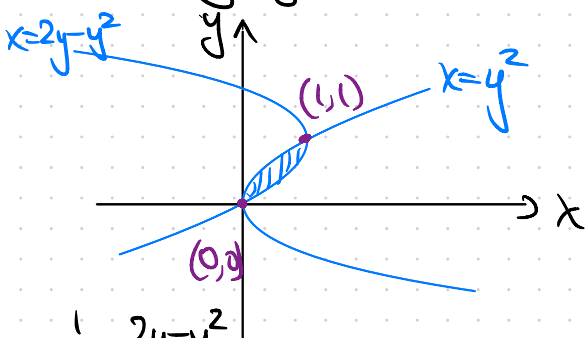
$$=\int_0^2(-x^2+4)\,dx+\int_0^4\sqrt{x}\,dx$$

$$=-\frac{1}{3}x^3\Big|_0^2+4x\Big|_0^2+\frac{2}{3}x^{\frac{3}{2}}\Big|_0^4$$

$$=-\frac{8}{3}+8+\frac{16}{3}=\boxed{\frac{32}{3}}$$

14.3 26) If $f(x,y) = 100(y+1)$ represents the population density of a planar region on Earth, where x and y are measured in kilometers, find the number of people in the region bounded by the curves $x=y^2$ and $x=2y-y^2$.

Sol'n:



Intersection: $y^2 = 2y - y^2 \Rightarrow 0 = 2y - 2y^2$

$0 = 2y(1-y)$

So $y = 0, 1$.

when $y=0$, $x=0$,

$y=1$, $x=1$.

$$\text{So } I = \int_0^1 \int_{y^2}^{2y-y^2} f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_{y^2}^{2y-y^2} 100(y+1)\,dx\,dy = \int_0^1 100(y+1)\,x \Big|_{x=y^2}^{x=2y-y^2} dy$$

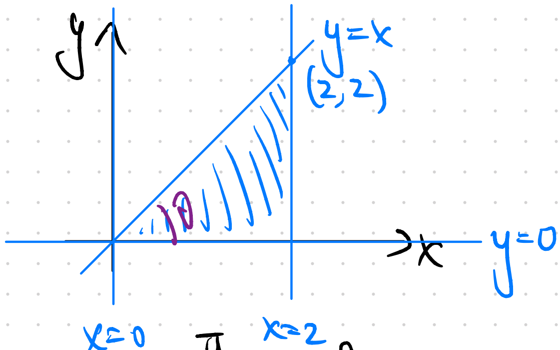
$$= \int_0^1 100(y+1)(2y-2y^2)\,dy = \int_0^1 (-200y^3+200y)\,dy$$

$$= -\frac{200}{4}y^4\Big|_0^1 + \frac{200}{2}y^2\Big|_0^1 = \frac{200}{2} - \frac{200}{4} = \boxed{50}$$

14.4 14) Change the Cartesian integral into an equivalent polar integral. Then evaluate the integral.

$$\int_0^2 \int_0^x y\,dy\,dx$$

Soln:



$0 \le x \le 2$ $\quad x = r\cos\theta$ $\quad 0 \le r\cos\theta \le 2$

$0 \le y \le x$ $\quad y = r\sin\theta$ $\quad 0 \le r\sin\theta \le r\cos\theta$

So in particular, $0 \le r \le 2\sec\theta$

$0 \le \tan\theta \le 1$

$\Updownarrow$

$0 \le \theta \le \frac{\pi}{4}$.

$$\int_0^2 \int_0^x y\,dy\,dx = \int_0^{\frac{\pi}{4}} \int_0^{2\sec\theta} r^2 \sin\theta \,dr\,d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{3} r^3 \Big|_0^{2\sec\theta} \sin\theta \,d\theta = \int_0^{\frac{\pi}{4}} \frac{8}{3}\sec^3\theta \sin\theta \,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{8}{3} \tan\theta \sec^2\theta \,d\theta = \frac{8}{3} \cdot \frac{1}{2} \tan^2\theta \Big|_0^{\frac{\pi}{4}} = \boxed{\frac{4}{3}}$$

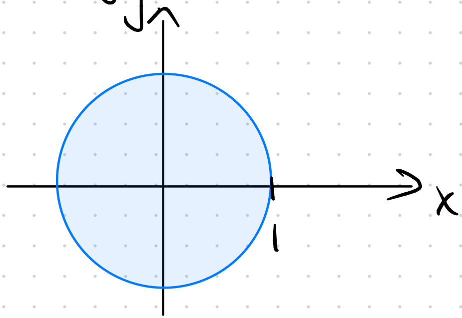
$\frac{d\tan\theta}{d\theta} = \sec^2\theta$

14.4 18) Change the Cartesian integral into an equivalent polar integral. Then evaluate the integral.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\,dy\,dx.$$

Sol'n: $x = r\cos\theta$
$y = r\sin\theta$

$dy\,dx \rightsquigarrow r\,dr\,d\theta.$ $\quad \dfrac{2}{(1+x^2+y^2)^2} = \dfrac{2}{(1+r^2)^2}$



$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\,dy\,dx = \int_{0}^{2\pi}\int_{0}^{1} \frac{2r}{(1+r^2)^2}\,dr\,d\theta$$

$u = 1+r^2$
$du = 2r\,dr$

$r=1 \Rightarrow u=2$
$r=0 \Rightarrow u=1$

$$= \int_{0}^{2\pi}\int_{1}^{2} \frac{1}{u^2}\,du\,d\theta$$

$$= \int_{0}^{2\pi} -\frac{1}{u}\Big|_{1}^{2}\,d\theta$$

$$= \int_0^{2\pi} \left(-\frac{1}{2} - (-1)\right) d\theta = \int_0^{2\pi} \frac{1}{2} d\theta = \frac{1}{2}\theta \Big|_0^{2\pi} = \boxed{\pi}$$